Reliability-based and imperfection-oriented structural optimization

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Summary

The real world behavior and the sensitiveness of structures are continuously gaining interest in structural design. This is a direct result of improvements due to advances made in modern computational mechanics. By that, the demands on structural optimization models representing an abstract and rational basis for computer-aided design also increase. To map reality, in particular, probabilistic phenomena are becoming essential and have to be taken into account. Within this contribution, two approaches are presented that allow for a realistic but simultaneously efficient reproduction of the probabilistic nature associated with structural design problems. First, a semi-deterministic approach is elucidated by which the random nature of geometric imperfections induced into a structural system can be caught. Second, a probabilistic approach is presented considering loads and stresses as random processes that lead to design quantities described by probability density functions.

Introduction

To solve a design problem in engineering it has proved to be a good idea to replace the original design problem by an equivalent optimization problem. In this context, the transformation from the design level to the optimization level requires the definition of three fundamental quantities: (i) optimization variables in terms of a vector \( x \), (ii) an optimization criterion or objective function \( f(x) \) and (iii) a set of constraints governed by the abstract inequalities \( g(x) \leq 0 \).

The confidence in the results of structural optimization depends to a large extent on the quality of the structural analysis method applied. In the majority of cases, the constraints defining the feasible region of the design space are dominated by structural analysis. Sometimes, the objective function may contain quantities that are directly computed from structural response (e.g. displacements or stresses). As a consequence, structural analysis pinpoints the unknown optimum. This makes the structure obtained by optimum design extremely sensitive to unpredictable effects. If the real world structure differs only very little from the solution achieved by the optimum design model, pathological phenonema may easily occur with inevitable consequences. It is therefore advisable to utilize structural models in the optimization process that are as precise as necessary, but also as efficient as possible. Efficiency, besides appropriate precision, is extraordinarily significant in numerical optimization because of the iterative nature of the solution process in which the evaluation of the given structure is repeated over and over. What is needed is a pertinent platform for comparing current designs with competitive design alternatives.

For the purpose of corroboration two application fields have been investigated in two separate research projects, funded by the Deutsche Forschungsgemeinschaft/DFG (German Research Association). In the first project, it is investigated which structural optimization model best qualifies geometric imperfections induced into a structure. It is demonstrated that the danger of geometric imperfections can be sufficiently encapsulated through a new method by which a random field problem is reduced to a deterministic one, based on a worst case assumption. In the second project, the ultimate limit and the serviceability limit states during the lifespan of a structure are of interest, taking e.g. the structural volume as the objective function. This requires constraints that incorporate time-variant deteriorations or damages to represent the structural reliability. It is assumed that deteriorations and damages are caused by random load processes which vary in time and create random stress fluctuations.
Fundamentals of imperfection-oriented optimization

Geometric imperfections can be described by the vector equation

$$\mathbf{x}_{\text{imp}} = \mathbf{x}_{\text{per}} + \mathbf{h}(\mathbf{x}_{\text{per}})$$

(1)

where the vectors $\mathbf{x}_{\text{imp}}$ and $\mathbf{x}_{\text{per}}$ define the location of the imperfect and perfect geometry of the structure, respectively, while the vector $\mathbf{h}(\mathbf{x}_{\text{per}})$ describes the offset of the perfect structure as a function of vector $\mathbf{x}_{\text{per}}$. The mapping function $\mathbf{h}$ is random due to the lapse of construction and, therefore, unknown at the time of design. Based on the theory of random fields, a spectral decomposition of $\mathbf{h}$ can be carried out [1]. Then, the function $\mathbf{h}$ can be represented by

$$\mathbf{h}(\mathbf{x}_{\text{per}}) = \sum_i y_i \mathbf{N}_i(\mathbf{x}_{\text{per}})$$

(2)

where $\mathbf{y}$ is a random vector of stochastically independent components $y_i$ and $\mathbf{N}_i(\mathbf{x}_{\text{per}})$ are base functions. Hence, the vector $\mathbf{x}_{\text{imp}}$ yields a random field. Correspondingly, the random nature of the structure geometry representation results in a random structural response in terms of displacements $\mathbf{u}$ such that

$$\mathbf{u} = \mathbf{u}(\mathbf{y}).$$

(3)

To reduce the random field problem to a deterministic one the following approach is carried out: Provided that imperfections are not arbitrarily large an envelope is introduced containing $P^*$ percentage (e. g. $P^* = 95\%$) of the potential shapes of imperfection. This can be achieved by choosing suitable base functions for the vector $\mathbf{h}(\mathbf{x}_{\text{per}})$ in Eqn. (1). To secure the $P^*$- quantile it is required that the relationship

$$P(\|\mathbf{y}\| \leq \varepsilon) = P^*$$

(4a)

holds. In Eqn. (4a) $P$ denotes the probability that the norm $\|\mathbf{y}\|$ being not greater then the quantity $\varepsilon$ equals the given value $P^*$. The quantity $\varepsilon$, determined and described by means of

$$\|\mathbf{y}\| \leq \varepsilon,$$

(4b)

represents the envelope of the expected imperfection shapes.

Based on this idea, a specific optimization problem is formulated as

$$\min_{\mathbf{x}} \max_{\mathbf{y}} \{f(\mathbf{x}, \mathbf{u}(\mathbf{x}, \mathbf{y}))| \mathbf{x} \in \mathcal{S}, \|\mathbf{y}\| \leq \varepsilon \}$$

(5)

where the feasible domain $\mathcal{S}$ is defined as

$$\mathcal{S} = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}_f \leq \mathbf{x} \leq \mathbf{x}_u, \mathbf{g}(\mathbf{x}, \mathbf{u}(\mathbf{x}, \mathbf{y})) \leq 0 \right\}.$$  

(6)

In Eqs. (5) and (6), again, the vector $\mathbf{x}$ encompasses the optimization variables $x_i, i=1,2,3,...,n$. In contrast to the generic formulation used in the introduction, the objective function $f(\mathbf{x})$ and the constraints $\mathbf{g}(\mathbf{x})\leq 0$ are now adapted to the structural mechanics model applied. This gives $f(\mathbf{x}) = f(\mathbf{x}, \mathbf{u}(\mathbf{x}, \mathbf{y}))$ and $\mathbf{g}(\mathbf{x}) = \mathbf{g}(\mathbf{x}, \mathbf{u}(\mathbf{x}, \mathbf{y})) \leq 0$. Furthermore, side constraints are introduced where $\mathbf{x}_f$ and $\mathbf{x}_u$ are lower and upper bounds, respectively, which, along with the constraints $\mathbf{g}(\mathbf{x}) = \mathbf{g}(\mathbf{x}, \mathbf{u}(\mathbf{x}, \mathbf{y})) \leq 0$, define the feasible domain $\mathcal{S}$ as subspace of the Euclidian space $\mathbb{R}^n$.

The solution of the Min-Max-optimization problem is denoted by the optimum design vector $\mathbf{x}^*$. This vector must be feasible and minimize the objective function $f$ while simultaneously creating the worst case shape of imperfection, depending on the quantity $\varepsilon$. According to the Min-Max-formulation the total optimization problem is partitioned into a two level problem (see Fig. 1). At the first level, the most disadvantageous shape of imperfection is computed for each current vector $\mathbf{x}$. Based upon this result, the structural optimization is carried out at the second level. Two nested loops, an internal and an external loop, provide the required solution $\mathbf{x}^*$. 

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Application examples for imperfection-oriented structural optimization

Considering a plane curved girder consisting of 129 truss bars interconnected at 61 nodes (see Fig. 2) the effectiveness of the above approach can be demonstrated. If non-uniform rational B-splines (NURBS) are applied for both the geometry of the structure and the imperfections the shape optimization problem can be described by virtue of 9 optimization variables $x_i, i=1,2,3,...,9$. In addition, the 3 sizing variables $x_{10}, x_{11} \text{ and } x_{12}$ are defined for the cross-sectional areas of the upper as well as lower chord and the diagonal bars of the girder. Furthermore, 5 uncertainty variables $y_i, i=1,2,3,4,5$ are introduced for the imperfections. The objective function $f(x, u(x,y))$ is specified by the sum of the absolute values of the nodal displacements.

Fig. 1: Min-Max optimization logic

Fig. 2: Girder to be optimized
such that
\[ f(x, u(x, y)) = \sum \|u_{NN}\|. \] (7)

Eqn. (7) characterizes the structural stiffness of the total system. The minimization of the function \( f \) is subject to two categories of constraints, first, a prescribed bound to the weight of the girder and second, bounds to the total height as well as width of the structure.

To illustrate the effects of the underlying structural analysis model with respect to optimization two distinct subproblems are evaluated. In the first subproblem, a girder is optimized using a perfect geometry and assuming small displacements as well as linear elasticity. The initial solution for this case is compared to the optimal solution in Fig. 2. A subsequent geometrically non-linear finite element analysis indicates that the optimized structure is completely useless, as shown in Fig. 3. At only 4\% of the loading a snap through of the structure occurs yielding a collapse load factor \( \lambda_c = 0.04 \).

![Fig. 3: Optimization of the linearly elastic behaving girder](image1)

In the second subproblem the optimization is carried out based upon the Min-Max-philosophy given above (see Fig. 1). In this case, a non-linear finite element analysis is already employed during the whole optimization. Due to the worst-case imperfection strategy used, for each created optimization vector \( x \) the most disadvantageous shape of imperfection is computed, in advance to each iteration step of the optimization. The corresponding optimal design for this approach is stable and depicted in Fig. 4.

![Fig. 4: Imperfection-oriented optimization of the girder](image2)
In contrast to the optimum based on a perfect structure (see Fig. 3) the imperfect solution is much more reasonable and acceptable.

**Fundamentals of a reliability-based structural optimization**

If the reliability during the lifespan of a structural system is the main focus of design it is mandatory to incorporate the stochastic nature of dynamic loadings. Load fluctuations in small as well as large spaces of time, are substantial for the dynamic response of a structure (e.g. in terms of time-variant displacements and stresses). This structural response ultimately causes deteriorations, accumulating to damages as a result of crack and fatigue phenomena. Conventionally, the computation of the short-term dynamic response is carried out by means of a time history method based upon the integration of the governing equation of motion. For structural optimization, this approach suffers from inefficiency. Thus, the covariance analysis has been applied using a finite element analysis along with a so-called „shaping-filter“ [2].

The filter models the stochastic nature of the load process. Considering e.g. wind loading as a colored noise signal the filter provides a reduction to white noise which is a prerequisite for a solution based upon the covariance method. In particular, the emerging Lyapunov matrix equation requires the existence of white noise excitation. This matrix equation results from a transformation of the state representation of the governing differential equation of motion, describing the vibration of the dynamically excited structure, in terms of

\[
\mathbf{M}\ddot{\mathbf{x}} + \mathbf{D}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{p}(t) .
\]

Eqn. (8) is linked to the shape-filter described by

\[
\dot{\mathbf{z}}_{sf}(t) = \mathbf{A}_{sf} \mathbf{z}_{sf}(t) + \mathbf{B}_{sf} \mathbf{w}(t),
\]

\[
\mathbf{p}(t) = \mathbf{C}_{sf} \mathbf{z}_{sf}(t) .
\]

In Eqn. (8), the matrices \(\mathbf{M}, \mathbf{D}\) and \(\mathbf{K}\) are the mass matrix, the damping matrix and the stiffness matrix, respectively. The vectors \(\ddot{\mathbf{x}}, \dot{\mathbf{x}}\) and \(\mathbf{x}\) are the acceleration vector, the velocity vector and the displacement vector and \(\mathbf{p}(t)\) is the load vector. The realization of the filter, which transforms the colored noise \(\mathbf{p}(t)\) into white noise \(\mathbf{w}(t)\), demands the creation of the three filter matrices \(\mathbf{A}_{sf}, \mathbf{B}_{sf}\) and \(\mathbf{C}_{sf}\), respectively. Connecting the matrix equation of motion with the filter equations leads to the state space equation in matrix form as follows

\[
\begin{pmatrix}
\dot{\mathbf{x}} \\
\ddot{\mathbf{x}} \\
\dot{\mathbf{z}}_{sf}
\end{pmatrix} =
\begin{bmatrix}
\mathbf{0} & \mathbf{1} & \mathbf{0} \\
\mathbf{0} & -\mathbf{M}^{-1}\mathbf{K} - \mathbf{M}^{-1}\mathbf{D} & \mathbf{M}^{-1}\mathbf{C}_{sf} \\
\mathbf{0} & \mathbf{0} & \mathbf{A}_{sf}
\end{bmatrix}
\begin{bmatrix}
\mathbf{x} \\
\dot{\mathbf{x}} \\
\mathbf{z}_{sf}
\end{bmatrix}
+ \begin{bmatrix}
\mathbf{0} \\
\dot{\mathbf{w}}(t) \\
\mathbf{B}_{sf}\mathbf{z}_{sf}
\end{bmatrix} = \mathbf{A}
\begin{pmatrix}
\dot{\mathbf{x}} \\
\ddot{\mathbf{x}} \\
\dot{\mathbf{z}}_{sf}
\end{pmatrix} + \mathbf{B} \mathbf{w}(t).
\]

From Eqn. (10) the Lyapunov matrix equation can be derived in terms of

\[
\mathbf{AP}_{zz} + \mathbf{P}_{zz} \mathbf{A}^T + \mathbf{BB}^T = 0 ,
\]

where \(\mathbf{P}_{zz}\) is the covariance matrix

\[
\mathbf{P}_{zz} = \int_{0}^{\infty} e^{\mathbf{A}^T t} \mathbf{BB}^T e^{\mathbf{A} t} dt .
\]

Evaluating the integral \(\mathbf{P}_{zz}\), after some intermediate computations, the even and odd spectral moments of the random stress response can be computed. Given these spectral moments, the expected frequency of zero-crossings \(N_0\) and of peaks per unit-time \(N_1\) can be deduced both of which allow the judging of the irregularity of the random process.
By using the numerical counting method of Dirlik [3], an appropriate approximation of the probability density function (PDF) for load induced stresses can also be established. The PDF serves as the starting point for computing the probability of failure \( p_f \) for a structural system due to stress fluctuations as

\[
p_f = \int_{D_f} \text{PDF}_{X_r}(x_r) \, dx_r ,
\]

where \( X_r \) is the vector of the random basic variables and \( x_r \) are the realizations of \( X_r \). The domain of integration \( D_f \) defines the region of failure.

**Application example for reliability-based structural optimization**

A case study for a framed steel structure (see Fig. 5) has been accomplished to exemplify the significance of probabilistic load and stress effects to structural optimization. Exclusively dynamic wind load fluctuations have been assumed to act upon the structure.

The objective function \( f \) is the cross-sectional area of the frame where an I-shaped profile is selected. For the purpose of simplicity, only two optimization variables, the height \( x_1 \) and the width \( x_2 \) of the cross section, are defined. Considered is the concrete problem of minimizing the objective function \( f(x_1, x_2) = f(x) \) subject to the side constraints

\[
0.1 \text{m} \leq x_1 \leq 1.0 \text{m}, \\
0.1 \text{m} \leq x_2 \leq 0.3 \text{m}, \tag{14}
\]

and to the constraint of failure probability \( p_f(x, x_r) \leq p_f,\text{accepted} \):

\[
g(x, x_r, p(t)) = p_f(x, x_r, p(t)) - p_f,\text{accepted} \leq 0 . \tag{15}
\]

Without going into the details of converting the constraint of failure probability into a numeric representation, the probability nature of the problem should be visualized. Fig. 5 depicts the characteristics of the reliability-based optimization. The geometrical representation of the minimization problem in the \( x_1 \) and \( x_2 \)-plane shows a contour plot of the function \( f \) and the „feasible“ region \( \mathcal{S} \) of the problem, described by means of the side constraints and the probabilistic constraint \( g(x, x_r, p(t)) \). As can be seen the constraint creates a scattered band according to the associated PDF. The scattering effect indicates that three categories of the cross-sections have to be distinguished. In the first category, all design candidates definitely fail. In the second, all candidates definitely survive. Between both categories there is a third scattering domain of design solutions which either fail or survive associated with a certain probability. By way of example, solutions may be included within a 90% or a 10%-quantile, respectively. To make sure that the optimum sought lies e.g. in the 99%-quantile an approach is in progress. To find the optimum (minimum) an evolution strategy (a search method based on optimization principles like mutation, re-combination, survival of the fittest, etc.) is applied. The circle in Fig. 5 labels the „optimum solution interval“.

![Probabilistic reliability constraint](image)

**Fig. 5: System layout and wind loading, plot of constraints and objective functions**
References

